# A note on the invariant subspace problem relative to a type $\mathrm{II}_1$ factor

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**Abstract**

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor with a faithful normal tracial state $\tau$ and let $\mathcal{M}^\omega$ be the ultrapower algebra of $\mathcal{M}$. In this paper, we prove that for every operator $T \in \mathcal{M}^\omega$, there is a family of projections $\{P_t\}_{0\leq t\leq 1}$ in $\mathcal{M}^\omega$ such that $TP_t = P_tTP_t$, $P_s \leq P_t$ if $s \leq t$, and $\tau_\omega(P_t) = t$. Let $\mathfrak{M} = \{Z \in \mathcal{M} :$ there is a family of projections $\{P_t\}_{0\leq t\leq 1}$ in $\mathcal{M}$ such that $ZP_t = P_tZP_t, P_s \leq P_t$ if $s \leq t$, and $\tau(P_t) = t\}$. As an application we show that for every operator $T \in \mathcal{M}$ and $\epsilon > 0$, there is an operator $S \in \mathfrak{M}$ such that $\|S\| \leq \|T\|$ and $\|S-T\|_2 < \epsilon$. We also show that $\prod_n^\omega M_n(\mathbb{C})$ is not $*$-isomorphic to the ultrapower algebra of the hyperfinite type $\mathrm{II}_1$ factor.

**Keywords:** Invariant subspaces, type $\mathrm{II}_1$ factors, ultrapower algebras.
**MSC:** 46L10, 47C15

## 1 Introduction

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor acting on a Hilbert space $\mathcal{H}$. The invariant subspace problem relative to a factor von Neumann algebra $\mathcal{M}$ asks for every operator $T \in \mathcal{M}$, does there exists a projection $P \in \mathcal{M}$, $0 < P < I$, such that $TP = PTP$. The hyperinvariant subspace problem relative to $\mathcal{M}$ asks for every operator $T \in \mathcal{M} \setminus \mathbb{C}I$, does there exists a projection $P$, $0 < P < I$, such that $SP = PSP$ for every operator $S$ in $\mathcal{B}(\mathcal{H})$ with $ST = TS$. It is easy to see that if a projection $P$ is hyperinvariant for $T$, then $P$ is in the von Neumann algebra generated by $T$ and therefore in $\mathcal{M}$. A huge advance on the (hyper)invariant subspace problem relative to a factor of type $\mathrm{II}_1$ has been made during past ten years (see for example [2, 3, 6, 13]).

In 1983, Brown [1] introduced a spectral distribution measure for non-normal elements in a finite von Neumann algebra with respect to a fixed normal faithful tracial state, which is called the Brown measure of the operator. Recently, Haagerup and Schultz [6] proved a remarkable result which states that if the support of Brown measure of an operator in a type $\mathrm{II}_1$ factor contains more than two points, then the operator has a non-trivial

hyperinvariant subspace affiliated with the type $\mathrm{II}_1$ factor. However, the invariant subspace problem relative to a type $\mathrm{II}_1$ factor still remains open for operators with single point Brown measure support (for this case, we refer to Dykema and Haagerup's paper [2]).

Suppose that each $\mathcal{M}_n$ is a finite von Neumann algebra with a faithful normal tracial state $\tau_n$. Let $\prod_{n\in\mathcal{N}} \mathcal{M}_n$ be the $l^\infty$-product of the $\mathcal{M}_n$'s. Then $\prod_n \mathcal{M}_n$ is a von Neumann algebra (with pointwise multiplication). Let $\omega$ be a free ultrafilter on $\mathcal{N}$ ($\omega$ may be viewed as an element in $\beta\mathcal{N}\setminus\mathcal{N}$, where $\beta\mathcal{N}$ is the Stone-Céch compactification of $\mathcal{N}$). If $\{X_n\}$ and $\{Y_n\}$ are two elements in $\prod_n \mathcal{M}_n$, then we define $\{X_n\}\sim\{Y_n\}$ when $\lim_{n\to\omega}\|X_n-Y_n\|_2=0$. Recall that for an operator $T_n\in\mathcal{M}_n$, $\|T_n\|_2=\tau_n(T_n^*T_n)^{1/2}$. Then the *ultraproduct*, denoted by $\prod^\omega \mathcal{M}_n$, of $\mathcal{M}_n$ (with respect to the free ultrafilter $\omega$) is the quotient von Neumann algebra of $\prod_n \mathcal{M}_n$ modulo the equivalence relation $\sim$ and the limit of $\tau_n$ at $\omega$ gives rise to a tracial state on $\prod^\omega \mathcal{M}_n$. We shall use $\tau_\omega$ to denote the tracial state on $\prod^\omega \mathcal{M}_n$. When $\mathcal{M}_n=\mathcal{M}$ for all $n$, then $\prod^\omega \mathcal{M}_n$ is called the *ultrapower* of $\mathcal{M}$, denoted by $\mathcal{M}^\omega$. The initial algebra $\mathcal{M}$ is embedded into $\mathcal{M}^\omega$ as constant sequences given by elements in $\mathcal{M}$. Ultrapowers for finite von Neumann algebras were first introduced and studied by McDuff [8]. Sakai [12] showed that an ultrapower of a finite von Neumann algebra with respect to a faithful normal trace is again a finite von Neumann algebra, and the ultrapower algebra $\mathcal{M}^\omega$ of a type $\mathrm{II}_1$ factor is also a type $\mathrm{II}_1$ factor. Ultrapowers of type $\mathrm{II}_1$ factors play an important role in the study of type $\mathrm{II}_1$ factors.

This paper is organized as follows. In section 2 of this paper, we prove that every operator in an ultrapower algebra of a type $\mathrm{II}_1$ factor $\mathcal{M}$ has a nontrivial invariant space affiliated with the ultrapower algebra. Precisely, we prove that for every operator $T\in\mathcal{M}^\omega$, there is a family of projections $\{P_t\}_{0\le t\le 1}$ in $\mathcal{M}^\omega$ such that $TP_t=P_tTP_t$, $P_s\le P_t$ if $s\le t$, and $\tau_\omega(P_t)=t$. This result is more or less trivial if $\mathcal{M}$ has property $\Gamma$. Recall that $\mathcal{M}$ is said to have property $\Gamma$ if for any finite elements $T_1,\cdots,T_n$ in $\mathcal{M}$ and $\epsilon>0$, there is a unitary operator $U$ in $\mathcal{M}$ such that $\tau(U)=0$ and $\|T_iU-UT_i\|_2<\epsilon$ for $1\le i\le n$. If $\mathcal{M}$ is a separable (with separable predual) type $\mathrm{II}_1$ factor, then $\mathcal{M}$ has property $\Gamma$ if and only if $\mathcal{M}'\cap\mathcal{M}^\omega$ is non-trivial. Dixmier [Di] proved that if $\mathcal{M}'\cap\mathcal{M}^\omega$ is non-trivial, then it is non-atomic. This implies that if $\mathcal{M}$ has property $\Gamma$, then for every operator $T\in\mathcal{M}^\omega$, there is a family of projections $\{P_t\}_{0\le t\le 1}$ in $\mathcal{M}^\omega$ such that $TP_t=P_tT$, $P_s\le P_t$ if $s\le t$, and $\tau_\omega(P_t)=t$. To prove the result for non-$\Gamma$ factors, we need combine techniques developed by Haagerup and Schultz [6] and a result of Popa [10].

As an application, in section 3 we show that for every operator $T$ in the unit ball of $\mathcal{M}$ and $\epsilon>0$, there is an operator $S\in\mathfrak{M}$ such that $\|S\|\le 1$ and $\|S-T\|_2<\epsilon$, where $\mathfrak{M}=\{Z\in\mathcal{M}:$ there is a family of projections $\{P_t\}_{0\le t\le 1}$ in $\mathcal{M}$ such that $ZP_t=P_tZP_t, P_s\le P_t$ if $s\le t$, and $\tau(P_t)=t\}$. In particular, this implies that $\mathfrak{M}$ is dense in $\mathcal{M}$ in the strong operator topology.

In section 4, we give a very simple proof of $\prod_n^\omega M_n(\mathbb{C})$ is not $*$-isomorphic to the ultrapower algebra of the hyperfinite type $\mathrm{II}_1$ factor (this result might be known to specialists, however we can not find it in the existed literature). This result relies on a result of Herrero and Szarek [5] (also see [17]).

Thanks to the existence of a faithful normal tracial state on a type $\mathrm{II}_1$ factor, in section 5 we show that if two operators $S$ and $T$ are quasi-similar in a type $\mathrm{II}_1$ factor $\mathcal{M}$, then $LatS \cap \mathcal{M}$ is not trivial if and only if $LatT \cap \mathcal{M}$ is not trivial. As a corollary, we show that for two operator $S, T$ in $\mathcal{M}$, $Lat(ST) \cap \mathcal{M}$ is not trivial if and only if $Lat(TS) \cap \mathcal{M}$ is not trivial. On the other hand, if the same result also holds for arbitrary two operators in $\mathcal{B}(\mathcal{H})$, then the answer to the classical invariant subspace problem is affirmative (see Remark 5.7).

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# 2 Invariant subspaces for operators in the ultrapower algebras

The main result of this section is the following result.

**Theorem 2.1.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $\mathcal{M}^\omega$ be the ultrapower algebra of $\mathcal{M}$. For every operator $T \in \mathcal{M}^\omega$, there is a family of projections $\{P_t\}_{0\le t\le 1}$ in $\mathcal{M}^\omega$ such that $TP_t = P_tTP_t$, $P_s \le P_t$ if $s \le t$, and $\tau_\omega(P_t) = t$.*

**Corollary 2.2.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor with a faithful normal tracial state $\tau$. For every operator $T \in \mathcal{M}$ and $0 \le t \le 1$, there is a sequence of projections $P_n \in \mathcal{M}$ such that $\lim_{n\to\infty} \|TP_n - P_nTP_n\|_2 = 0$ and $\tau(P_n) = t$.*

To prove Theorem 2.1, we need the following lemmas.

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $T \in \mathcal{M}$. We regard $\mathcal{M}$ as a subfactor of $\mathcal{M}_1 = \mathcal{M} * Ł(\mathbb{F}_4)$. The faithful normal tracial state on $\mathcal{M}_1$ will also be denoted by $\tau$. We choose a circular system $\{x, y\}$ (in the sense of [16]) that generates $Ł(\mathbb{F}_4)$ and which therefore is free from $\mathcal{M}$. By Theorem 5.2 of [7], the unbounded operator $z = xy^{-1}$ is in $L^p(\mathcal{M}_1, \tau)$ for $0 < p < 1$. Let $T_n = T + \frac{1}{n}z$. Then $T_n \in L^p(\mathcal{M}_1, \tau)$ for $0 < p < 1$. We will need the following lemma, which follows from Proposition 4.5, Corollary 4.6, Theorem 5.1 and Theorem 6.9 of [6].

**Lemma 2.3.** *With the above assumption, we have*

*1.* $\lim_{n\to\infty} \|T - T_n\|_p^p = 0$*;*

*2. for every* $n$*, there is a projection* $P_n \in \mathcal{M}_1$ *such that* $T_nP_n = P_nT_nP_n$ *and* $\tau(P_n) = \frac{1}{2}$*.*

The next lemma follows from the main theorem of [10].

**Lemma 2.4.** *Let* $\mathcal{M}$ *be a separable type* $\mathrm{II}_1$ *factor. Then there is a unitary operator* $u \in \mathcal{M}^\omega$ *such that*

$$\{\mathcal{M}, u\mathcal{M}u^*\}'' \cong \mathcal{M} * (u\mathcal{M}u^*).$$

**Lemma 2.5.** *Let* $\mathcal{M}$ *be a separable type* $\mathrm{II}_1$ *factor and let* $T \in \mathcal{M}$*. Then for every* $\epsilon > 0$*, there is a projection* $P \in \mathcal{M}$*,* $\tau(P) = 1/2$*, such that* $\|TP - PTP\|_2 < \epsilon$*.*

*Proof.* Note that $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{M}^\omega$ if we identify $T \in \mathcal{M}$ with the constant sequence $(T) \in \mathcal{M}^\omega$. To prove the lemma, it is sufficient to show that there is a projection $P \in \mathcal{M}^\omega$, $\tau(P) = 1/2$, such that $\|TP - PTP\|_2 < \epsilon$. By Lemma 2.4, there is a unitary operator $u \in \mathcal{M}^\omega$ such that $\{\mathcal{M}, u\mathcal{M}u^*\}'' \cong \mathcal{M}*(u\mathcal{M}u^*)$. So it is sufficient to show that there is a projection $P \in \{\mathcal{M}, u\mathcal{M}u^*\}''$, $\tau(P) = 1/2$, such that $\|TP - PTP\|_2 < \epsilon$. Note that $T \in \mathcal{M}$ and therefore $T$ is free with $u\mathcal{M}u^*$ in $\{\mathcal{M}, u\mathcal{M}u^*\}''$. Repeat the above arguments twice if necessary, we may assume that $\mathcal{M} \supseteq Ł(\mathbb{F}_4)$ and $T$ is free with $Ł(\mathbb{F}_4)$.

We choose a circular system $\{x, y\}$ in $Ł(\mathbb{F}_4)$. Let $z = xy^{-1}$ and $T_n = T + \frac{1}{n}z$. By Lemma 4.2, for every $n \geq 1$, there is a projection $P_n \in \mathcal{M}$ with $\tau(P_n) = 1/2$ and $T_nP_n = P_nT_nP_n$. By Lemma 4.2, $\lim_{n\to\infty} \|T_n - T\|_p^p = 0$ for $0 < p < 1$. Note that

$$\begin{aligned}
\|P_nTP_n - TP_n\|_2^2 &= \tau(|P_nTP_n - TP_n|^2) \\
&= \tau(|P_nTP_n - TP_n|^{p/2}|P_nTP_n - TP_n|^{2-p/2}) \\
&\leq \tau(|P_nTP_n - TP_n|^p)^{1/2}\tau(|P_nTP_n - TP_n|^{4-p})^{1/2} \\
&= \|P_nTP_n - TP_n\|_p^{p/2}\|P_nTP_n - TP_n\|_{4-p}^{(4-p)/2} \\
&\leq \left(\|P_nTP_n - TP_n\|_p^p\right)^{1/2} \|2T\|_{4-p}^{(4-p)/2},
\end{aligned}$$

and

$$\begin{aligned}
\|P_nTP_n - TP_n\|_p^p &\leq \|P_n(T - T_n)P_n - (T - T_n)P_n\|_p^p \\
&\leq \|P_n(T - T_n)P_n\|_p^p + (T - T_n)P_n\|_p^p \\
&\leq 2\|T - T_n\|_p^p \to 0.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|P_nTP_n - TP_n\|_2^2 = 0$.

□

**Lemma 2.6.** *Let $\mathcal{M}$ be a separable type* $\mathrm{II}_1$ *factor,* $T \in \mathcal{M}$ *and* $\epsilon > 0$*. For every positive integer* $n$*, there are projections* $\{P_j\}_{j=0}^{2^n}$ *in* $\mathcal{M}$ *such that* $0 = P_0 < P_1 < P_2 < \cdots < P_{2^n-1} < P_{2^n} = I$*,* $\tau(P_j) = j/2^n$*, and* $\|TP_j - P_jTP_j\|_2 \leq \epsilon$ *for all* $0 \leq j \leq 2^n$*.*

*Proof.* If $n = 1$, then the lemma follows from Lemma 2.5. Suppose $n = 2$. By Lemma 2.5, there are projections $P, Q$ in $\mathcal{M}$ such that $\tau(P) = \tau(Q) = 1/2$, $P + Q = 1$ and $\|TP - PTP\|_2 < \epsilon/2$. Let $a = PTP$, $b = PTQ$, $c = QTP$, and $d = QTQ$. We can write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the decomposition $I = P + Q$. Then $\|c\|_2 < \epsilon/2$. Note that both $P\mathcal{M}P$ and $Q\mathcal{M}Q$ are type $\mathrm{II}_1$ factors. We apply Lemma 2.5 to $a \in P\mathcal{M}P$ and $b \in Q\mathcal{M}Q$, respectively. There are projections $P_1 \leq P$, $Q_1 \leq Q$ such that $\tau(P_1) = \tau(Q_1) = 1/4$, $\|aP_1 - P_1aP_1\|_2 < \epsilon/2$ and $\|bQ_1 - Q_1bQ_1\|_2 < \epsilon/2$. Let $P_0 = 0$, $P_2 = P$, $P_3 = P + Q_1$, and $P_4 = I$. Then $0 = P_0 < P_1 < P_2 < P_3 < P_4 = I$ and $\tau(P_j) = j/4$ for $0 \leq j \leq 4$. Simple computations show that $\|TP_j - P_jTP_j\|_2 \leq \epsilon$ for all $0 \leq j \leq 4$. The general case can be proved by using the induction on $n$ with similar arguments as the above. □

Combining Lemma 2.6 and the noncommutative Hölder's inequality, we have the following:

**Corollary 2.7.** *Let $\mathcal{M}$ be a separable type* $\mathrm{II}_1$ *factor and let* $T \in \mathcal{M}$*. Then for every* $\epsilon > 0$ *and every* $t$ *with* $0 \leq t \leq 1$*, there is a projection* $P \in \mathcal{M}$*,* $\tau(P) = t$*, such that* $\|TP - PTP\|_2 < \epsilon$*.*

The following lemma extends Lemma 2.5 to arbitrary type $\mathrm{II}_1$ factors.

**Lemma 2.8.** *Let $\mathcal{M}$ be a type* $\mathrm{II}_1$ *factor and let* $T \in \mathcal{M}$*. Then for every* $\epsilon > 0$*, there is a projection* $P \in \mathcal{M}$*,* $\tau(P) = 1/2$*, such that* $\|TP - PTP\|_2 < \epsilon$*.*

*Proof.* Let $\mathcal{N}$ be the von Neumann subalgebra generated by $T$. Then $\mathcal{N}$ is separable. If $\mathcal{N}' \cap \mathcal{M}$ is a diffuse von Neumann algebra, then for every $t$, $0 \leq t \leq 1$, there is a projection $P \in \mathcal{N}' \cap \mathcal{M}$ such that $PT = TP$ and $\tau(P) = t$. Hence Lemma 2.8 follows. If $\mathcal{N}' \cap \mathcal{M}$ is not a diffuse von Neumann algebra, let $P_0, P_1, P_2, \cdots$ be a sequence of projections in $\mathcal{N}' \cap \mathcal{M}$ such that $P_0 + P_1 + P_2 + \cdots = I$, $P_0(\mathcal{N}' \cap \mathcal{M})P_0$ is diffuse, and $P_1, P_2, \cdots$ are non-zero minimal projections in $(1 - P_0)(\mathcal{N}' \cap \mathcal{M})(1 - P_0)$. Note that $(\mathcal{N}P_n)' \cap (P_n\mathcal{M}P_n) = P_n(\mathcal{N}' \cap \mathcal{M})P_n = \mathbb{C}P_n$ for $n \geq 1$. This implies that $\mathcal{N}P_n$ is a separable type $\mathrm{II}_1$ factor for $n \geq 1$. There is an $n \geq 0$ such that $\sum_{k=1}^{n} \tau(P_k) \leq t \leq \sum_{k=1}^{n+1} \tau(P_k)$. Applying Corollary 2.7 to $\mathcal{N}P_{n+1}$, $t' = t - \sum_{k=1}^{n} \tau(P_k)$, and $TP_{n+1}$, there is a projection $Q_{n+1} \in \mathcal{N}P_{n+1}$ such that $\tau(Q_{n+1}) = t'$ and

$$\|TP_{n+1}Q_{n+1} - Q_{n+1}TP_{n+1}Q_{n+1}\|_2 < \epsilon.$$

Let $P = P_0 + P_1 + \cdots + P_n + Q_{n+1}$. Then $P \in \mathcal{M}$, $\tau(P) = t$, and

$$\|TP - PTP\|_2 < \epsilon.$$

$\square$

As a consequence of Lemma 2.8, Lemma 2.6 is also true for arbitrary type $\mathrm{II}_1$ factors.

*Proof of Theorem 2.1.* Let $T = (T_n) \in \mathcal{M}^\omega$. By Lemma 2.6, for each $n$, there are projections $\{P_{n,j}\}_{0\le j\le 2^n}$ in $\mathcal{M}$ such that $0 = P_{n,0} < P_{n,1} < P_{n,2} < \cdots < P_{n,2^n-1} < P_{n,2^n} = I$, $\tau(P_{n,j}) = j/2^n$, and $\|T_nP_{n,j} - P_{n,j}T_nP_{n,j}\|_2 \le 1/n$ for all $0 \le j \le 2^n$. For every $t$, $0 \le t \le 1$, choose $P_{n,j}$ such that $\tau(P_{n,j}) \le t < \tau(P_{n,j+1})$. Let $P_t = (P_{n,j}) \in \mathcal{M}^\omega$. Then $P_s \le P_t$ if $s \le t$, $\tau_\omega(P_t) = t$, and $TP_t = P_tTP_t$.

$\square$

# 3 Operators with non-trivial invariant subspaces relative to a type $\mathrm{II}_1$ factor

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor with a faithful normal tracial state $\tau$, and let $\mathfrak{M} = \{S' \in \mathcal{M} :$ there is a family of projections $\{P_t\}_{0\le t\le 1}$ in $\mathcal{M}$ such that $ZP_t = P_tZP_t, P_s \le P_t$ if $s \le t$, and $\tau(P_t) = t\}$. Let $(\mathcal{M})_1$ be the set of operators $T$ in $\mathcal{M}$ such that $\|T\| \le 1$. As an application of Theorem 2.1, we prove the following result.

**Theorem 3.1.** *For every operator $T \in (\mathcal{M})_1$ and every $\epsilon > 0$, there is an operator $S \in \mathfrak{M} \cap (\mathcal{M})_1$ such that $\|T - S\|_2 < \epsilon$. In particular, the set $\mathfrak{M}$ is dense in $\mathcal{M}$ in the strong operator topology.*

To prove Theorem 3.1, we need the following lemmas. The following lemma is well known.

**Lemma 3.2.** *Suppose $\{T_n\}_n \subseteq (\mathcal{M})_1$ is a Cauchy sequence with respect to $\|\cdot\|_2$. Then there is an operator $T \in (\mathcal{M})_1$ such that*

$$\lim_{n\to\infty} \|T_n - T\|_2 = 0.$$

For an operator $T \in \mathcal{M}$, let $N(T)$ be the projection onto the kernel space of $T$.

**Lemma 3.3.** *Let $\epsilon, \delta > 0$ and $T \in \mathcal{M}$. If $\|T\|_2 < \delta$, then there is a projection $P \in \mathcal{M}$ such that $P \ge N(T)$, $\|TP\| \le \epsilon$, and $\tau(I - P) < \delta^2/\epsilon^2$.*

*Proof.* By applying the polar decomposition theorem, we may assume that $T$ is a positive operator. Let $\nu$ be the Borel measure on $[0,\infty)$ induced by the composition of $\tau$ with the spectral projections of $T$. Then

$$\|T\|_2^2 = \int_0^\infty t^2 d\nu(t) < \delta^2.$$

Let $P = \chi_{[0,\epsilon]}(T)$. Then $P \geq N(T)$, $\|TP\| \leq \epsilon$ and

$$\epsilon^2 \tau(I-P) \leq \int_\epsilon^\infty t^2 d\nu(t) \leq \|T\|_2^2 < \delta^2.$$

Hence, $\tau(I-P) < \delta^2/\epsilon^2$. □

**Lemma 3.4.** *For every operator $T \in (\mathcal{M})_1$ and every $\epsilon > 0$, there is an operator $S \in (\mathcal{M})_1$ such that*

1. $\|T-S\|_2 < \epsilon$ *and*
2. *there is a projection $P \in \mathcal{M}$ such that $\tau(P) = 1/2$ and $SP = PSP$.*

*Proof.* Choose $\delta, \epsilon_1 > 0$ such that

$$\epsilon_1 + \epsilon_1/\delta + \delta < \epsilon.$$

By Corollary 2.2, there is a projection $P_1$ in $\mathcal{M}$ such that

$$\|TP_1 - P_1TP_1\|_2 < \delta. \tag{3.1}$$

Let $P_2 = I - P_1$ and $T_{ij} = P_iTP_j$ for $i,j = 1,2$. Then we can write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with respect to the decomposition $I = P_1 + P_2$. Since $\|T\| \leq 1$, $\|T_{ij}\| \leq 1$ for all $i,j = 1,2$. Note that ( 3.1) implies $\|T_{21}\|_2 < \delta$ and also note that $N(T_{2,1}) \geq P_2$. By Lemma 3.3, there is a projection $Q \in \mathcal{M}$, $Q \geq P_2$, $\|T_{21}Q\| \leq \epsilon_1$ and $\tau(I-Q) < \epsilon_1^2/\delta^2$. Write $Q = P_1' + P_2$. Then $P_1' \leq P_1$ and $\tau(P_1 - P_1') < \epsilon_1^2/\delta^2$.

Let $R = T_{11}P_1' + T_{12} + T_{22}$, i.e., we can write

$$R = \begin{pmatrix} T_{11}P_1' & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

with respect to the decomposition $I = P_1 + P_2$. Then $R - TQ = T_{21}Q$. Therefore,

$$\|R\| = \|TQ + T_{21}Q\| \leq 1 + \epsilon_1. \tag{3.2}$$

On the other hand, $R - T = T_{11}(P_1 - P_1') + T_{21}$. This implies that

$$\|R - T\|_2 \leq \|T_{11}(P_1 - P_1')\|_2 + \|T_{21}\|_2 \leq \epsilon_1/\delta + \delta. \tag{3.3}$$

Let $S = (1 + \epsilon_1)^{-1}R$. Then ( 3.2) implies that $\|S\| \leq 1$ and (3.3) implies that

$$\|S - T\|_2 \leq \|S - R\|_2 + \|R - T\|_2 \leq \epsilon_1\|S\|_2 + \epsilon_1/\delta + \delta \leq \epsilon_1 + \epsilon_1/\delta + \delta < \epsilon.$$

Note that $SP_1 = P_1SP_1$ and $\tau(P_1) = 1/2$. Let $P = P_1$. We prove the lemma.

□

*Proof of Theorem 3.1.* We use the induction to construct operators $T_n$ and $\{P_{n,j}\}_{j=1}^{2^n}$ for each $n \geq 0$ satisfying the following conditions:

1. for each $n$, $\{P_{n,j}\}_{j=1}^{2^n}$ is a family of projections in $\mathcal{M}$ such that $\sum_{j=1}^{2^n} P_{n,j} = I$ and $\tau(P_{n,j}) = 1/2^n$ for $1 \leq j \leq 2^n$;
2. $P_{n,j} = P_{n+1,2j-1} + P_{n+1,2j}$ for $1 \leq j \leq 2^n$;
3. $\|T_n\| \leq 1$, $T_0 = T$, and $\|T_n - T_{n+1}\|_2 < \epsilon/2^{n+1}$;
4. for each $k$, $1 \leq k \leq 2^n$, $\sum_{j=1}^{k} P_{n,j}$ is an invariant subspace of $T_n$.

For $n = 0$, let $T_0 = T$ and $P_{0,1} = I$. For $n = 1$, by Lemma 3.4, there is an operator $S \in \mathcal{M}$, $\|S\| \leq 1$, $\|S - T\|_2 < \epsilon/2$ and there is a projection $P \in \mathcal{M}$, $\tau(P) = 1/2$ and $SP = PSP$. Let $T_1 = S$, $P_{1,1} = P$ and $P_{1,2} = I - P$. Now for $n = 2$, we construct $T_2$ and $\{P_{2,j}\}_{j=1}^{4}$ satisfying the above conditions 1,2,3 and 4.

Since $P_{1,1}$ is an invariant subspace of $T_1$, we can write

$$T_1 = \begin{pmatrix} A & T_{12} \\ 0 & B \end{pmatrix}$$

with respect to the decomposition $I = P_{1,1} + P_{1,2}$. Let $\epsilon_1, \delta > 0$ such that

$$\epsilon_1 + 3\epsilon_1/\delta + 2\delta < \epsilon/4.$$

Applying Corollary 2.2 to $A \in P_{1,1}\mathcal{M}P_{1,1}$ and $B \in P_{1,2}\mathcal{M}P_{1,2}$, there are projections $Q_1, Q_2, Q_3, Q_4$ such that $\tau(Q_j) = 1/4$ for $1 \le j \le 4$, $Q_1 + Q_2 = P_{1,1}$, $Q_3 + Q_4 = P_{1,2}$, $\|AQ_1 - Q_1AQ_1\|_2 < \delta$ and $\|BQ_3 - Q_3BQ_3\|_2 < \delta$. Now we can write

$$T_1 = \begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} & T_{12} \\ 0 & \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{pmatrix}$$

with respect to the decomposition $I = Q_1 + Q_2 + Q_3 + Q_4$. Note that $\|AQ_1 - Q_1AQ_1\|_2 < \delta$ implies $\|A_{21}\|_2 < \delta$ and $\|BQ_3 - Q_3BQ_3\|_2 < \delta$ implies $\|B_{21}\|_2 < \delta$. By Lemma 3.3 and similar arguments as the proof of Lemma 3.4, there are projections $Q_1' \le Q_1$, $Q_3' \le Q_3$ such that $\|A_{21}Q_1'\| < \epsilon_1$, $\|B_{21}Q_3'\| < \epsilon_1$, $\tau(Q_1 - Q_1') \le \epsilon_1^2/\delta^2$ and $\tau(Q_3 - Q_3') \le \epsilon_1^2/\delta^2$.

Let

$$R = \begin{pmatrix} \begin{pmatrix} A_{11}Q_1' & A_{12} \\ 0 & A_{22} \end{pmatrix} & T_{12}(Q_3' + Q_4) \\ 0 & \begin{pmatrix} B_{11}Q_3' & B_{12} \\ 0 & B_{22} \end{pmatrix} \end{pmatrix}$$

with respect to the decomposition $I = Q_1 + Q_2 + Q_3 + Q_4$. Then

$$\|R - T_1(Q_1' + Q_2 + Q_3' + Q_4)\| = \|A_{21}Q_1' + B_{21}Q_3'\| < \epsilon_1$$

and

$$\|R - T_1\|_2 = \|A_{11}(Q_1 - Q_1') + A_{21} + T_{12}(Q_3 - Q_3') + B_{11}(Q_3 - Q_3') + B_{21}\|_2 \le 3\epsilon_1/\delta + 2\delta.$$

Therefore,

$$\|R\| \le \|T_1(Q_1' + Q_2 + Q_3' + Q_4)\| + \|R - T_1(Q_1' + Q_2 + Q_3' + Q_4)\| < 1 + \epsilon_1.$$

Let $T_2 = (1 + \epsilon_1)^{-1}R$. Then $\|T_2\| \le 1$ and

$$\|T_2 - T_1\|_2 \le \|T_2 - R\|_2 + \|R - T_1\|_2 < \epsilon_1\|T_2\| + 3\epsilon_1/\delta + 2\delta < \epsilon_1 + 3\epsilon_1/\delta + 2\delta < \epsilon/4.$$

Let $P_{2,j} = Q_j$ for $1 \le j \le 4$. Then $T_2$ and $\{P_{2,j}\}_{j=1}^4$ satisfy the conditions 1,2,3 and 4. The general case can be proved similarly by using the induction.

Suppose $T_n$ and $\{P_{n,j}\}_{j=1}^{2^n}$ satisfy the above conditions 1,2,3 and 4. By 3 and Lemma 3.2, there is an operator $S \in (\mathcal{M})_1$ such that $\lim_{n\to\infty} \|S - T_n\|_2 = 0$ and $\|S - T\|_2 < \epsilon$. By

2 and 4, for each $n$ and $k$, $1 \leq k \leq 2^n$, $\sum_{j=1}^{k} P_{n,j}$ is an invariant subspace of $T_N$ for $N \geq n$ and therefore an invariant subspace of $S$. By 1, $\tau(\sum_{j=1}^{k} P_{n,j}) = k/2^n$. Note that $\{k/2^n : n \geq 0, 1 \leq k \leq 2^n\}$ is dense in $[0,1]$. For every $t$, $0 \leq t \leq 1$, let

$$P_t = \bigvee_{k/2^n \leq t} \left( \sum_{j=1}^{k} P_{n,j} \right).$$

By 1, $P_s \leq P_t$ if $s \leq t$, $\tau(P_t) = t$ and $SP_t = P_t S P_t$.

□

# 4 $\prod^{\omega} M_n(\mathbb{C})$ is not $*$-isomorphic to $\mathcal{R}^{\omega}$

Throughout this section $\mathcal{M}$ is a separable type II$_1$ factor. Recall that a separable type II$_1$ factor $\mathcal{M}$ has property $\Gamma$ if for every $n$, $T_1, \cdots, T_n \in \mathcal{M}$, and every $\epsilon > 0$, there is a projection $P \in \mathcal{M}$ such that $\tau(P) = 1/2$ and $\|T_i P - P T_i\|_2 < \epsilon$ (cf. [4]).

**Lemma 4.1.** *Suppose $\mathcal{M}$ has property $\Gamma$. Then for every operator $T \in \mathcal{M}^{\omega}$ and $t$, $0 \leq t \leq 1$, there is a projection $P \in \mathcal{M}^{\omega}$ such that $PT = TP$ and $\tau_{\omega}(P) = 1/2$.*

*Proof.* Write $T = (T_n)$. Since $\mathcal{M}$ has property $\Gamma$, there exists a projection $P_n \in \mathcal{M}$ such that $\|P_n T_n - T_n P_n\|_2 < 1/n$ and $\tau(P_n) = 1/2$. Let $P = (P_n) \in \mathcal{M}^{\omega}$. Then $PT = TP$ and $\tau_{\omega}(P_n) = 1/2$. □

Let $(M_n(\mathbb{C}))_1$ be the set of matrices $T \in M_n(\mathbb{C})$ such that $\|T\| \leq 1$, and let $\nu((M_n(\mathbb{C}))_1, \omega)$ be the covering number of $(M_n(\mathbb{C}))_1$ with respect to the normalized trace norm $\|\cdot\|_2$. There are universal constants $c_1, c_2$ [14, 15] such that

$$\left(\frac{c_1}{\omega}\right)^{2n^2} \leq \nu((M_n(\mathbb{C}))_1, \omega) \leq \left(\frac{c_2}{\omega}\right)^{2n^2}. \tag{4.1}$$

The next lemma follows from Theorem 9 of Herrero and Szarek [5] (also see [17]). For the sake of completeness, we include a direct proof.

**Lemma 4.2.** *There exists a universal constant $\alpha > 0$ with the following property: for each $n \geq 2$, there exists a matrix $T_n \in M_n(\mathbb{C})$, $\|T_n\| = 1$, such that*

$$\|PT_n - T_n P\|_2 \geq \alpha$$

*for every projection $P \in M_n(\mathbb{C})$ with* $\mathrm{rank} P = \left[\frac{n}{2}\right]$, *where $\left[\frac{n}{2}\right]$ is the maximal integer less or equal to $\frac{n}{2}$.*

*Proof.* Suppose the lemma is false. Then for every $\epsilon > 0$, there is an $n \geq 2$, for every matrix $T \in M_n(\mathbb{C})$, $\|T\| \leq 1$, there is a projection $P \in M_n(\mathbb{C})$ such that $\text{rank}P = \left[\frac{n}{2}\right]$ and $\|PT - TP\|_2 < \epsilon$. Without of loss of generality we may assume that $n = 2k$. Let $(M_n(\mathbb{C}))_1$ be the set of $n \times n$ complex matrices $T$ such that $\|T\| \leq 1$. For $T \in M_n(\mathbb{C})$, let $\|T\|_2$ be the trace norm with respect to the normalized trace $\tau_n = \frac{Tr}{n}$ on $M_n(\mathbb{C})$.

By (4.1),

$$\left(\frac{c_1}{2\epsilon}\right)^{2n^2} \leq \nu((M_n(\mathbb{C}))_1, 2\epsilon) \leq \left(\frac{c_2}{2\epsilon}\right)^{2n^2} \tag{4.2}$$

and

$$\left(\frac{c_1}{\epsilon}\right)^{2k^2} \leq \nu((M_k(\mathbb{C}))_1, \epsilon) \leq \left(\frac{c_2}{\epsilon}\right)^{2k^2}.$$

Let $\{T_t\}_{t\in\mathbb{T}}$ be an $\epsilon$-net of $(M_k(\mathbb{C}))_1$ such that $\#\mathbb{T} \leq \left(\frac{c_2}{\epsilon}\right)$.

Now for every $T \in (M_n(\mathbb{C}))_1$, $\|TP - PT\|_2 < \epsilon$ for some projection $P \in M_n(\mathbb{C})$ with rank $k$. Write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

with respect to the decomposition $I = P + (I - P)$. Since $\|T\| \leq 1$, $\|T_{11}\|, \|T_{22}\| \leq 1$. Choose $t_1, t_2 \in \mathbb{T}$ such that $\|T_{11} - T_{t_1}\|_2 < \epsilon$ and $\|T_{22} - T_{t_2}\|_2 < \epsilon$ with respect to the normalized trace norm on $M_k(\mathbb{C})$. Since $\|TP - PT\|_2 < \epsilon$,

$$\|T - \begin{pmatrix} T_{t_1} & 0 \\ 0 & T_{t_2} \end{pmatrix}\|_2 < 2\epsilon.$$

This implies that,

$$\nu((M_n(\mathbb{C}))_1, 2\epsilon) \leq \left(\frac{c_2}{\epsilon}\right)^{2k^2} \cdot \left(\frac{c_2}{\epsilon}\right)^{2k^2} = \left(\frac{c_2}{\epsilon}\right)^{4k^2}. \tag{4.3}$$

Note that $n = 2k$. By (4.2),

$$\left(\frac{c_1}{2\epsilon}\right)^{2n^2} \leq \left(\frac{c_2}{\epsilon}\right)^{n^2}.$$

By taking ln on both sides, we have

$$\frac{2(\ln c_1 - \ln 2 - \ln \epsilon)}{-\ln \epsilon} \leq \frac{\ln c_2 - \ln \epsilon}{-\ln \epsilon}.$$

Let $\epsilon \to 0+$. This implies $2 \leq 1$. This is a contradiction. $\square$

**Theorem 4.3.** *The von Neumann algebra $\prod^{\omega} M_n(\mathbb{C})$ is not $*$-isomorphic to $\mathcal{R}^{\omega}$, the ultrapower algebra of the hyperfinite* $\mathrm{II}_1$ *factor.*

*Proof.* Choose $T_n \in M_n(\mathbb{C})$ as in Lemma 4.2. Let $T = (T_n) \in \prod^{\omega} M_n(\mathbb{C})$. Claim if $P$ is a projection in $\prod^{\omega} M_n(\mathbb{C})$ such that $TP = PT$, then $\tau_{\omega}(P) \neq 1/2$. Otherwise, suppose $P = (P_n) \in \prod^{\omega} M_n(\mathbb{C})$ is a projection such that $TP = PT$ and $\tau_{\omega}(P) = 1/2$. We may assume that $P_n$ is a projection in $M_n(\mathbb{C})$ with $\mathrm{rank} P = \left[\frac{n}{2}\right]$. By Lemma 4.2, $\|T_nP_n - P_nT_n\|_2 \geq \alpha > 0$. Hence $\|PT - TP\|_2 \geq \alpha > 0$. This is a contradiction. On the other hand, for every operator $T \in \mathcal{R}^{\omega}$, there is a projection $Q \in \mathcal{R}^{\omega}$ such that $TQ = QT$ and $\tau_{\omega}(Q) = 1/2$ by Lemma 4.1. So $\prod^{\omega} M_n(\mathbb{C})$ is not $*$-isomorphic to $\mathcal{R}^{\omega}$. $\square$

**Remark 4.4.** By Theorem 9 of [5], there is an operator $T$ in $\prod^{\omega} M_n(\mathbb{C})$ such that if $TP = PT$ for some projection $P$ in $\prod^{\omega} M_n(\mathbb{C})$, then $P = 0$ or $P = I$.

**Question:** Can $\mathcal{R}^{\omega}$ be embedded into $\prod^{\omega} M_n(\mathbb{C})$? If $\mathcal{M}$ is a separable type $\mathrm{II}_1$ factor and $\mathcal{M}^{\omega} \cong \mathcal{R}^{\omega}$, is $\mathcal{M} \cong \mathcal{R}$?

# 5 The lattice of invariant subspaces of an operator affiliated with a type $\mathrm{II}_1$ factor

Let $\mathcal{M}$ be a factor (not necessarily type $\mathrm{II}_1$) acting on a Hilbert space $\mathcal{H}$ and $T \in \mathcal{M}$. We denote by $Lat_{\mathcal{M}}T$ the set of projections $P \in \mathcal{M}$ such that $TP = PTP$. So $P \in Lat_{\mathcal{M}}$ if and only if $P\mathcal{H}$ is an invariant subspace of $T$. Recall that a hyperinvariant subspace of $T$ is a (closed) subspace invariant under every operator in $\{T\}'$. It is easy to see that the projection onto a hyperinvariant subspace of $T$ is in the von Neumann algebra generated by $T$.

Suppose $S, T$ are two operators in $\mathcal{M}$. Recall that $S$ and $T$ are *quasi-similar* in $\mathcal{M}$ if there are operators $X, Y \in \mathcal{M}$ which are one-to-one and have dense range such that $SX = XT$ and $YS = TY$. The following theorem is given in [11](Theorem 6.19).

**Theorem 5.1.** *If $S$ and $T$ are quasi-similar in $\mathcal{B}(\mathcal{H})$ and $S$ has a nontrivial hyperinvariant subspace, then $T$ has a nontrivial hyperinvariant subspace.*

It is still not known that if we replace the hyperinvariant subspace by the invariant subspace in the above theorem, the theorem still holds or not. However, in this section we will show that if we replace $\mathcal{B}(\mathcal{H})$ by a type $\mathrm{II}_1$ factor and replace the hyperinvariant subspace by the invariant subspace, then the above theorem still holds.

We denote by $N(T)$ the kernel space of $T$ and $R(T)$ the closure of range space of $T$.

**Lemma 5.2.** *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace $\tau$, and let $T \in \mathcal{M}$. Then $\tau(R(T)) + \tau(N(T)) = 1$. In particular, $N(T) = 0$ if and only if $R(T) = I$.*

*Proof.* By the polar decomposition theorem, there is a unitary operator $U$ and a positive operator $|T|$ in $\mathcal{M}$ such that $T = U|T|$. So $T^* = |T|U^*$. Now, we have $T^*T = |T|^2 = U^*TT^*U$. Thus, $\tau(R(T)) = \tau(R(TT^*)) = \tau(R(T^*T)) = \tau(R(T^*)) = 1 - \tau(N(T))$. $\square$

**Corollary 5.3.** *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace $\tau$. Let $T \in \mathcal{M}$ be an operator such that $N(T) = 0$, and let $E \in \mathcal{M}$ be a projection. Then $\tau(R(TE)) = \tau(E)$. In particular, if $0 < E < I$, then $0 < R(TE) < I$.*

*Proof.* Since $N(T) = 0$, $N(TE) = I - E$. By lemma 5.2, $\tau(R(TE)) = 1 - \tau(N(TE)) = 1 - \tau(I - E) = \tau(E)$. $\square$

**Proposition 5.4.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor with a faithful normal trace $\tau$ and $S, T \in \mathcal{M}$. If there is an operator $X \in \mathcal{M}$ such that $N(X) = 0$ and $XS = TX$, then $LatS$ is isomorphic to a sublattice of $LatT$ and $LatT$ is isomorphic to a sublattice of $LatS$. In particular, $S$ has a nontrivial invariant subspace if and only if $T$ has a nontrivial invariant subspace.*

*Proof.* For $E \in Lat_{\mathcal{M}}S$, let $F = R(XE)$. The assumption $XS = TX$ implies that $F \in Lat_{\mathcal{M}}T$. Define $\phi(E) = F$. By corollary 5.3, $\tau(F) = \tau(E)$. We want to show that $\phi$ is a lattice isomorphism from $Lat_{\mathcal{M}}S$ onto a sublattice of $Lat_{\mathcal{M}}T$. Let $E_1, E_2 \in LatS$. Then $\phi(E_1 \vee E_2) = R(X(E_1 \vee E_2)) = R(XE_1) \vee R(XE_2) = \phi(E_1) \vee \phi(E_2)$ and $\phi(E_1 \wedge E_2) = R(X(E_1 \wedge E_2)) \leq R(X(E_1)) \wedge R(X(E_2)) = \phi(E_1) \wedge \phi(E_2)$. By corollary 5.3,

$$\tau(\phi(E_1) \wedge \phi(E_2)) = \tau(\phi(E_1) \vee \phi(E_2)) - \tau(\phi(E_1)) - \tau(\phi(E_2))$$

$$= \tau(E_1 \vee E_2) - \tau(E_1) - \tau(E_2) = \tau(E_1 \wedge E_2) = \tau(\phi(E_1 \wedge E_2)).$$

So $\phi(E_1 \wedge E_2) = \phi(E_1) \wedge \phi(E_2)$. Thus $\phi$ is a lattice homomorphism. Let $E_1, E_2 \in LatS$ and $E_1 \neq E_2$. We may assume that $E = E_1 \vee E_2 > E_1$. So $\tau(E) > \tau(E_1)$. If $\phi(E_1) = \phi(E_2) = F \in LatT$. Then $F = \phi(E_1 \vee E_2)$. By corollary 5.3, $\tau(F) = \tau(E_1) = \tau(E_1 \vee E_2) = \tau(E)$. This is a contradiction. So $\phi$ is a lattice isomorphism from $Lat_{\mathcal{M}}S$ onto a sublattice of $Lat_{\mathcal{M}}T$.

Similarly, by $X^*T^* = S^*X^*$, there is a lattice isomorphism from $Lat_{\mathcal{M}}T^*$ onto a sublattice of $Lat_{\mathcal{M}}S^*$. Since $Lat_{\mathcal{M}}T$ is isomorphism to $Lat_{\mathcal{M}}T^*$ and $Lat_{\mathcal{M}}S$ is isomorphic to $Lat_{\mathcal{M}}S^*$. So there is a lattice isomorphic from $Lat_{\mathcal{M}}T$ onto a sublattice of $Lat_{\mathcal{M}}S$. $\square$

**Proposition 5.5.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and $S, T \in \mathcal{M}$. If $S$ and $T$ are quasi-similar, then the lattice of hyperinvariant subspaces of $S$ and the lattice of hyperinvariant subspaces of $T$ are isomorphic.*

*Proof.* Let $X, Y$ in $\mathcal{M}$ be one to one operators with dense ranges such that $XS = TX$ and $SY = YT$. Let $E$ be a hyperinvariant subspace of $S$. Let $F$ the closure of the linear span of $R(AXE)$, where $AT = TA$. Then clearly $F$ is a hyperinvariant subspace of $T$. Note that $\tau(F) \geq \tau(XE) = \tau(E)$ by corollary 5.3. Since $YAXS = YATX = YTAX = SYAX$ and $E$ is a hyperinvariant subspace of $S$, $R(YAXE) \leq E$ and therefore, $R(YF) \leq E$. By corollary 5.3, $\tau(E) \geq \tau(F)$. So $\tau(F) = \tau(E)$, $F = R(XE)$, and $E = R(YF)$. Now $E \to F = R(XE)$ is a lattice isomorphism (the inverse is $F \to E = R(YF)$) from the lattice of hyperinvariant subspaces of $S$ onto the lattice of hyperinvariant subspaces of $T$. □

**Corollary 5.6.** *Let $\mathcal{M}$ be a type* $\mathrm{II}_1$ *factor and $S, T \in \mathcal{M}$. Then $Lat_{\mathcal{M}}ST$ is not trivial iff $Lat_{\mathcal{M}}TS$ is not trivial. Furthermore, if $N(S) = N(T) = 0$, then $Lat_{\mathcal{M}}ST$ is isomorphic to $Lat_{\mathcal{M}}TS$ and the lattice of hyperinvariant subspaces of $ST$ is isomorphic to the lattice of hyperinvariant subspaces of $TS$ as lattices.*

*Proof.* Suppose $Lat_{\mathcal{M}}ST$ is not trivial. If $TS = 0$, then $Lat_{\mathcal{M}}TS$ is not trivial. We assume that $TS \neq 0$. If $N(S) \neq 0$ or $R(T) \neq I$, then $N(S)$ or $R(T)$ is a non trivial invariant subspace of $TS$. if $N(S) = 0$ and $R(T) = I$, then by lemma 5.2, $R(S) = I$ and $N(T) = 0$. Thus $ST, TS$ are quasisimilar. By Proposition 5.4, $Lat_{\mathcal{M}}TS$ is not trivial.

If $N(S) = N(T) = 0$, then $R(S) = R(T) = I$ by lemma 5.2. For $E \in Lat_{\mathcal{M}}ST$, let $F = R(TE)$ and $E_1 = R(SF)$. Then $E_1 = R(SF) = R(STE) \leq E$ since $E \in Lat_{\mathcal{M}}ST$. By corollary 5.3, $\tau(E) = \tau(F) = \tau(E_1)$. This implies that $E = E_1$. Note that $R(TSF) = R(TSTE) \leq R(TE) = F$, $F \in Lat_{\mathcal{M}}TS$. Define $\phi(E) = R(TE)$ and $\psi(F) = R(SF)$ for $E \in Lat_{\mathcal{M}}ST$ and $F \in Lat_{\mathcal{M}}TS$, respectively. Then $\psi = \phi^{-1}$. So $\phi$ is a lattice isomorphism from $Lat_{\mathcal{M}}ST$ onto $Lat_{\mathcal{M}}TS$.

The lattice of hyperinvariant subspaces of $ST$ is isomorphic to the lattice of hyperinvariant subspaces of $TS$ as lattices is a corollary of Proposition 5.5. □

**Remark 5.7.** Let $T \in \mathcal{B}(H)$ and $V \in \mathcal{B}(H)$ such that $VV^* = I$ but $V^*V \neq I$. Then $R(V^*)$ is a nontrivial invariant subspace of $V^*TV$. Note that $T = TVV^*$. If the first part of Corollary 5.6 is true for $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then the answer to the invariant subspace question (relative to $\mathcal{B}(\mathcal{H})$ is affirmative.

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